Taylor's Frozen-in Hypothesis for Magnetohydrodynamic turbulence and Solar Wind

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In hydrodynamics, Taylor's frozen-in hypothesis connects the wavenumber spectrum to the frequency spectrum of a time series measured in real space. In this paper, we generalize Taylor's hypothesis to magnetohydrodynamic turbulence. We analytically derive one-point two-time correlation functions for Elsässer variables whose Fourier transform yields the corresponding frequency spectra, $E^{\pm}(f)$. We show that $E^{\pm}(f) \propto |\mathbf{U}_0 \mp \mathbf{B}_0|^{2/3}$ in Kolmogorov-like model, and $E^{\pm}(f) \propto (B_0|\mathbf{U}_0 \mp \mathbf{B}_0|)^{1/2}$ in Iroshnikov-Kraichnan model, where $\mathbf{U}_0, \mathbf{B}_0$ are the mean velocity and mean magnetic fields respectively.

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I. INTRODUCTION

Theoretical, experimental, and computational tools are employed for studying turbulence. Kolmogorov[1,2] constructed one of the most popular model of turbulence and showed that for homogeneous and isotropic turbulence, the kinetic energy spectrum in wavenumber space is

$$E_u(k) = K_{\text{Ko}}\epsilon_u^{2/3}k^{-5/3}, \tag{1}$$

where k is the wavenumber, ϵ_u is the dissipation rate of kinetic energy, and K_{Ko} is Kolmogorov's constant. Computation of $E_u(k)$ requires three-dimensional velocity field, which is difficult to measure in experiment. Instead, the velocity field is measured at select real-space points. Taylor[3] proposed an important conjecture that helps connect $E_u(k)$ to the frequency spectrum, $E_u(f)$, of the measured time series. Taylor[3] hypothesized that turbulent fluctuations are advected by the mean flow (velocity $\mathbf{U}_0$) as if fluctuations are frozen in the flow. Under this assumption, using Kolmogorov's spectrum of Eq. (1), we can easily derive the frequency spectrum as

$$E_u(f) = A(\epsilon_u U_0)^{2/3}f^{-5/3}, \tag{2}$$

where A is a constant. Thus, Taylor's hypothesis plays an important role in turbulence experiments and data analysis.

Kraichnan[4] argued that the large-scale eddies too sweep the small-scale fluctuations, a phenomenon called *sweeping effect*. Wilczek and Narita[5] and He et al.[6] generalized Eq. (2) by taking into account the sweeping effect. Recently, Verma et al.[7] performed a detailed calculation of temporal correlation function that includes sweeping effect and turbulent diffusion; they showed that $E_u(f) \propto f^{-5/3}$ in the presence of a mean flow, but $E_u(f) \propto f^{-2}$ when U_0 is small compared to the fluctuations.

In magnetohydrodynamic (MHD) turbulence, the velocity and magnetic fields have their mean values ($\mathbf{U}_0$ and $\mathbf{B}_0$, respectively, in velocity units) and fluctuations. Hence, Taylor's hypothesis and Eq. (2) need generalization, especially because Solar wind and solar corona are important MHD turbulence laboratories where spacecrafts make in situ measurements of the velocity and magnetic fields. Taylor's hypothesis would help us derive the wavenumber spectrum using the frequency spectrum of the time series. Note that the typical speed of spacecrafts are much smaller than the solar wind speed (250 km/s to 750 km/s). Hence,

the spacecraft can be assumed to be stationary, consistent with the assumptions of Taylor's hypothesis. Another important factor is that $B_0 \ll U_0$ at distances 0.3 AU or higher, where most spacecrafts are located. As a result, we can ignore the effects of the mean magnetic field and employ Taylor's hypothesis. In one of the first applications of Taylor's hypothesis to the solar wind, Matthaeus and Goldstein[8] deduced $k^{-5/3}$ spectrum for the solar wind by employing Eq. (2). There are many more works on solar wind observations, e.g., Podesta et al.[9].

Recent spacecraft *Parker Solar Probe* (PSP) has come quite close the Sun (as close as 8.86 solar radii). At this distance, B_0 is comparable to U_0, and hence, we need to examine whether Taylor's hypothesis can be employed to PSP data. In addition, the solar wind is anisotropic that makes an application of Taylor's hypothesis problematic. Bourouaine and Perez[10] modelled the Fourier-transformed temporal correlation function of MHD turbulence using a Gaussian model rather than a pure exponential model, and argued that the decorrelation frequency is linearly related to the perpendicular wavenumber. In a related work, Perez et al.[11] investigated the validity of Taylor's hypothesis for the PSP data, and showed that the frequency spectrum accurately represents the spectral indices associated with the underlying spatial spectrum of turbulent fluctuations in the plasma frame. In a another recent work, Kasper et al.[12] computed the frequency spectrum of the PSP data recorded at 13 million km above the photosphere, and observed the spectral index to be closer to $-3/2$ than $-5/3$. Note that Kasper et al.[12] assume Taylor's hypothesis for the solar wind even though B_0 and U_0 are comparable in this regime.

In this paper, we extend the derivation of Verma et al.[7] for hydrodynamic turbulence to MHD turbulence. We express the "dressed" or "renormalized" temporal correlation functions for the Elsässer variables in the Fourier space. Then, an inverse Fourier transform of the above correlation function yields two-point two-time correlation function. By setting the distance between the two points to zero, we deduce one-point two-time correlation functions, whose Fourier transform yields the frequency spectra for MHD turbulence. For Kolmogorov's and Iroshnikov-Kraichnan models[13,14], we obtain respectively $f^{-5/3}$ and $f^{-3/2}$ frequency spectra with prefactors that are functions of $|\mathbf{U}_0 \mp \mathbf{B}_0|$, which are the wave speeds of Alfvén waves. Thus, we generalize Taylor's hypothesis to MHD turbulence; our derivation is mathematically more rigorous than earlier ones.

In the next two sections, we derive the correlation functions for linear and turbulent

MHD.

II. CORRELATION FUNCTIONS FOR LINEAR MHD

The equations of a magnetofluid moving with a mean velocity of $\mathbf{U}_0$ in a background of a mean magnetic field $\mathbf{B}_0$ are

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{U}_0 \cdot \nabla)\mathbf{u} - (\mathbf{B}_0 \cdot \nabla)\mathbf{b} + (\mathbf{u} \cdot \nabla)\mathbf{u} - (\mathbf{b} \cdot \nabla)\mathbf{b} = -\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{f}, \tag{3}$$

$$\frac{\partial \mathbf{b}}{\partial t} + (\mathbf{U}_0 \cdot \nabla)\mathbf{b} - (\mathbf{B}_0 \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{b} - (\mathbf{b} \cdot \nabla)\mathbf{u} = \eta\nabla^2\mathbf{b}, \tag{4}$$

$$\nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \mathbf{b} = 0, \tag{5}$$

where $\mathbf{u}, \mathbf{b}$ are the velocity and magnetic field fluctuations, $\mathbf{f}$ is the external force, p is the pressure, ν is the kinematic viscosity, and η is the magnetic diffusivity. In the above equations, the magnetic field is velocity units, which is obtained by a transformation, $\mathbf{B}_{\mathrm{CGS}} \to \mathbf{B}_{\mathrm{CGS}}/\sqrt{4\pi\rho}$, where ρ is the material density of the flow. In this paper, we assume that the flow is incompressible.

Alfvén waves are the basic modes of linearized MHD equations, and they are conveniently expressed in terms of Elsässer variables, $\mathbf{z}^\pm = \mathbf{u} \pm \mathbf{b}$. Using Eqs. (3, 4, 5), we can derive the following equations for $\mathbf{z}^\pm$:

$$\frac{\partial \mathbf{z}^\pm}{\partial t} + \mathbf{Z}_0^\mp \cdot \nabla\mathbf{z}^\pm + (\mathbf{z}^\mp \cdot \nabla)\mathbf{z}^\pm = -\nabla p + \nu_\pm\nabla^2\mathbf{z}^\pm + \nu_\mp\nabla^2\mathbf{z}^\mp + \mathbf{f}, \tag{6}$$

$$\nabla \cdot \mathbf{z}^\pm = 0, \tag{7}$$

where $\mathbf{Z}_0^\mp = (\mathbf{U}_0 \mp \mathbf{B}_0)$, and $\nu_\pm = (\nu \pm \eta)/2$. For simplification, in this paper, we take $\nu = \eta$ that leads to $\nu_+ = \nu = \eta$ and $\nu_- = 0$. In Fourier space, the equations for $\mathbf{z}^\pm$ are

$$\left[\frac{\partial}{\partial t} + i\mathbf{Z}_0^\mp \cdot \mathbf{k} + \nu k^2\right] \mathbf{z}^\pm(\mathbf{k}) = -i\mathbf{k}p(\mathbf{k}) - i\sum_\mathbf{p}[\mathbf{k} \cdot \mathbf{z}^\mp(\mathbf{q})]\mathbf{z}^\pm(\mathbf{p}) + \mathbf{f}(\mathbf{k}), \tag{8}$$

$$\mathbf{k} \cdot \mathbf{z}^\pm(\mathbf{k}) = 0, \tag{9}$$

with $\mathbf{q} = \mathbf{k} - \mathbf{p}$. A linearized version of the above equations indicates that the waves $\mathbf{z}^+(\mathbf{k})$ and $\mathbf{z}^-(\mathbf{k})$ move with speeds $(\mathbf{U}_0 - \mathbf{B}_0) \cdot \mathbf{k}$ and $(\mathbf{U}_0 + \mathbf{B}_0) \cdot \mathbf{k}$ respectively.

Using the linearised versions of Eq. (8), we derive the equations for the Green's functions

$$\left[\frac{\partial}{\partial t} + i\mathbf{Z}_0^\mp \cdot \mathbf{k} + \nu k^2\right] G^\pm(\mathbf{k}, t, t') = \delta(t - t'), \tag{10}$$

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whose solutions are

$$G^{\pm}(\mathbf{k}, \tau) = \theta(\tau) \exp\left[-i\mathbf{Z}_0^{\mp} \cdot \mathbf{k}\tau\right] \exp\left(-\nu k^2 \tau\right), \tag{11}$$

where $\tau = t - t'$, and $\theta(\tau)$ is the step function.

The equal-time correction functions, $C^{\pm}(\mathbf{k}, 0)$, and two-time correction functions, $C^{\pm}(\mathbf{k}, \tau)$, for $\mathbf{z}^{\pm}(\mathbf{k})$ are defined as

$$C^{\pm}(\mathbf{k}, 0) = \langle |\mathbf{z}^{\pm}(\mathbf{k}, t)|^2 \rangle, \tag{12}$$

$$C^{\pm}(\mathbf{k}, \tau) = \langle \mathbf{z}^{\pm}(\mathbf{k}, t) \cdot \mathbf{z}^{\pm*}(\mathbf{k}, t + \tau) \rangle. \tag{13}$$

Note that $C^{\pm}(\mathbf{k}, \tau)$ is a complex function. We define the normalised correlation function as

$$R^{\pm}(\mathbf{k}, \tau) = \frac{C^{\pm}(\mathbf{k}, \tau)}{C^{\pm}(\mathbf{k}, 0)}. \tag{14}$$

A generalisation of fluctuation-dissipation theorem to MHD yields[15]

$$R^{\pm}(\mathbf{k}, \tau) = G^{\pm}(\mathbf{k}, \tau) = \theta(\tau) \exp\left(-i\mathbf{Z}_0^{\mp} \cdot \mathbf{k}\tau\right) \exp\left(-\nu k^2 \tau\right). \tag{15}$$

The above equation indicates that the normalised correlation functions exhibit damped oscillations.

III.   CORRELATION FUNCTIONS FOR TURBULENT MHD

A magnetofluid becomes turbulent when $UL/\nu \gg 1$ and $UL/\eta \gg 1$, where U, L are the large-scale velocity and length respectively. There is no definitive theory of MHD turbulence, rather it has many models[16–18]. In this paper we will focus on two leading models. Here, we focus on the shell spectra: $E^{\pm}(k)$, $E_u(k)$, $E_b(k)$, which are defined as

$$E^{\pm} = \frac{1}{2}\langle |\mathbf{z}^{\pm}|^2 \rangle = \int E^{\pm}(k)dk, \tag{16}$$

$$E_u = \frac{1}{2}\langle |\mathbf{u}^2| \rangle = \int E_u(k)dk, \tag{17}$$

$$E_b = \frac{1}{2}\langle |\mathbf{b}^2| \rangle = \int E_u(k)dk, \tag{18}$$

where $E^{\pm}, E_u, E_b$ are the total energies per unit volume of $\mathbf{z}^{\pm}$, $\mathbf{u}$, and $\mathbf{b}$ respectively.

1. *Komogorov-like MHD turbulence phenomenology*: In this framework, the energy spectra $E^{\pm}(k)$ are modelled as[16,19,20]

$$E^{\pm}(k) = K^{\pm}(\epsilon^{\pm})^{4/3}(\epsilon^{\mp})^{-2/3}k^{-5/3}, \tag{19}$$

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where $\epsilon^{\pm}$ are the inertial-range energy fluxes or dissipation rates of $\mathbf{z}^{\pm}$, and $K^{\pm}$ are constants, similar to Kolmogorov's constant for hydrodynamic turbulence. This phenomenology is also referred to as *imbalanced MHD*.

In addition, Goldreich and Sridhar[21] constructed a phenomenology for the anisotropic MHD turbulence. Using *critical balance* between the time scales for the nonlinear interactions and Alfvén wave propagation, they showed that the modal energy

$$\tilde{E}(k_{\perp}, k_{\parallel}) = K\epsilon^{2/3}k_{\perp}^{-10/3}g(k_{\parallel}/k_{\perp}^{2/3}), \tag{20}$$

where K is a constant, ϵ is the total dissipation rate, and $k_{\parallel}$ and $k_{\perp}$ are respectively the wavenumber components parallel and perpendicular to the mean magnetic field. Note that

$$\int k_{\perp}dk_{\perp}dk_{\parallel}\tilde{E}(k_{\perp}, k_{\parallel}) = E, \tag{21}$$

where E is the total energy.

2. *Iroshnikov-Kraichnan phenomenology*[13,14]: In this framework, the Alfvén time scale, $(kB_0)^{-1}$, is the relevant time scale, leading to the energy spectrum as

$$E_u(k) \approx E_b(k) \approx K_{\mathrm{IK}}(\epsilon B_0)^{1/2}k^{-3/2}, \tag{22}$$

where K_{IK} is constant, and B_0 is the amplitude of the mean magnetic field or that of large-scale magnetic field. In this phenomenology, the kinetic and magnetic energies are equipartitioned.

Dobrowolny et al.[22] showed that

$$\epsilon^{+} = \epsilon^{-} = \frac{1}{B_0}E^{+}(k)E^{-}(k)k^3. \tag{23}$$

For a special case when $E^{+}(k) = E^{-}(k) = E(k)$ (E is the total energy), we obtain

$$E^{+}(k) = E^{-}(k) = E(k) = K'(\epsilon B_0)^{1/2}k^{-3/2}. \tag{24}$$

Now, we model the correlation function for MHD turbulence following the strategies adopted for the hydrodynamic turbulence. The most critical part is the convective component. Using Eq. (8), we deduce the convective component to be $\exp\left(-i\mathbf{Z}_0^{\mp} \cdot \mathbf{k}\tau\right)$ for $\mathbf{z}^{\pm}$. In the following discussion we show that the convective part contributes most significantly to the frequency spectra.

The other two parts of the correlation function are the effective diffusion parameters and the sweeping effect. In hydrodynamic turbulence, field-theoretic treatment shows that the *renormalized viscosity* or *effective viscosity* ($\nu(k)$) is

$$\nu(k) = \nu_* \epsilon^{1/3} k^{-4/3}, \tag{25}$$

where ν_* is a nondimensional constant[16,23,24]. However, MHD turbulence has two diffusion parameters, viscosity and magnetic diffusivity, that depend on the cross helicity, Alfvén ratio, and mean magnetic field. We do not yet have general formulas for these renormalized parameters, even though they have been solved for special cases (see Verma[25,26,27]). In this paper, we simplify the calculation by assuming that both the renormalized parameters are equal (i.e., $\nu(k) = \eta(k)$), and that for Kolmogorov-like phenomenology,

$$\nu^{\pm}(k) = \nu_*^{\pm}(\epsilon^{\pm})^{1/3} k^{-4/3}, \tag{26}$$

and for Iroshnikov-Kraichnan phenomenology,

$$\nu^{\pm}(k) = \nu_*'(\epsilon B_0)^{1/4} k^{-5/4}. \tag{27}$$

Here, $\nu_*^{\pm}$ and ν_*' are constants. As we show in the next section, the terms with $\nu^{\pm}(k)$ get integrated in $E^{\pm}(f)$. Hence, a precise form of $\nu^{\pm}(k)$ may not be critical for the derivation of $E^{\pm}(f)$.

In addition, according to the *sweeping effect*, large-scale flow structures sweep the inertial-range fluctuations. For hydrodynamic turbulence, Kraichnan[4], Wilczek and Narita[5], Verma et al.[7] and others have constructed models for the sweeping effect. For MHD turbulence, we follow the prescription of Verma et al.[7] who added a random large-scale velocity field, $\tilde{\mathbf{U}}_0$, to the mean velocity field $\mathbf{U}_0$. These corrections are added in the correlation function for the linear equation [Eq. (15)].

Under the above assumptions, we arrive at the following expressions for the correlation functions of MHD turbulence[7]:

$$R^{\pm}(\mathbf{k}, \tau) = \frac{C^{\pm}(\mathbf{k}, \tau)}{C^{\pm}(\mathbf{k})} = \exp\left(-i\mathbf{Z}_0^{\mp} \cdot \mathbf{k}\tau\right) \exp(-i\tilde{\mathbf{U}}_0 \cdot \mathbf{k}\tau) \exp[-\nu^{\pm}(k)k^2\tau]. \tag{28}$$

Note that the correlations functions depend on both $\mathbf{U}_0$ and $\mathbf{B}_0$. In the next section, we will relate the above functions to Taylor's hypothesis for MHD turbulence.

IV. TAYLOR'S HYPOTHESIS FOR MHD TURBULENCE

Using Eq. (28), we first derive the two-point two-time correlation functions, after that we derive one-point two-time correlation functions, whose Fourier transform yields the frequency spectra of real-space time series of $\mathbf{z}^\pm$.

Using Eq. (28), we derive the following two-point two-time correlation functions for $\mathbf{z}^\pm$:

$$C^\pm(\mathbf{r}, \tau) = \int d\mathbf{k} C^\pm(\mathbf{k}) \exp[-\nu(k)k^2\tau - i\mathbf{Z_0^\mp} \cdot \mathbf{k}\tau] \exp[-i\mathbf{k} \cdot \tilde{\mathbf{U}}_0(\mathbf{k})\tau] \exp(i\mathbf{k} \cdot \mathbf{r}), \quad (29)$$

where $\mathbf{Z_0^\mp} = \mathbf{U}_0 \mp \mathbf{B}_0$. We ensemble average $C^\pm(\mathbf{r}, \tau)$ for random $\tilde{\mathbf{U}}_0$ (assuming isotropic, as in Kraichnan[4]) that yields[4,5,7]

$$C^\pm(\mathbf{r}, \tau) = \int d\mathbf{k} C^\pm(\mathbf{k}) \exp[-\nu^\pm(k)k^2\tau - i\mathbf{Z_0^\pm} \cdot \mathbf{k}\tau]\langle\exp[-ick\tilde{U}_0(k)\tau]\rangle \exp(i\mathbf{k} \cdot \mathbf{r})$$

$$= \int d\mathbf{k} C^\pm(\mathbf{k}) \exp[-\nu^\pm(k)k^2\tau - i\mathbf{Z_0^\pm} \cdot \mathbf{k}\tau] \exp[-c^2k^2\{\tilde{U}_0(k)\}^2\tau^2] \exp(i\mathbf{k} \cdot \mathbf{r}).(30)$$

For simplicity, we assume that the constant $c \approx 1$, and set $\mathbf{r} = 0$ to compute one-point two-time correlation functions $C^\pm(\mathbf{r} = 0, \tau) = C^\pm(\tau)$. The Gaussian model for the sweeping effect has been reported earlier by Kraichnan[4], Wilczek and Narita[5], and Bourouaine and Perez[10].

Now we derive $C^\pm(\tau)$ for the Kolmogorov-like phenomenology. Following Pope[28], we take $C^\pm(\mathbf{k})$:

$$C^\pm(\mathbf{k}) = \frac{E^\pm(k)}{2\pi k^2} = \frac{1}{2\pi k^2} f_L(kL) f_\eta(k\eta) K^\pm k^{-5/3} \frac{(\epsilon^\pm)^{4/3}}{(\epsilon^\mp)^{2/3}} \quad (31)$$

where

$$f_L(kL) = \left(\frac{kL}{[(kL)^2 + c_L]^{1/2}}\right)^{5/3+p_0}, \quad (32)$$

$$f_\eta(k\eta) = \exp\left[-\beta\left\{[(k\eta)^4 + c_\eta^4]^{1/4} - c_\eta\right\}\right] \quad (33)$$

are respectively the forcing and dissipative components of the energy spectra, and c_L, c_η, p_0, β are constants. We employ Eq. (26) for $\nu(k)$, and $\tilde{U}_0(k) = \epsilon^{1/3}k^{-1/3}$ with ϵ as the total dissipation rate (see Verma et al.[7]). In addition, we ignore the constants for brevity. After the above substitutions in Eq. (30) with $\mathbf{r} = 0$, we obtain

$$C^\pm(\tau) = K^\pm \frac{(\epsilon^\pm)^{4/3}}{(\epsilon^\mp)^{2/3}} \int dk k^{-5/3} f_L(kL) f_\eta(k\eta) \exp(-i\mathbf{Z_0^\mp} \cdot \mathbf{k}\tau) \times$$

$$\exp[-(\epsilon^\pm)^{1/3}k^{2/3}\tau] \exp[-\epsilon^{2/3}k^{4/3}\tau^2]. \quad (34)$$

The above integral is quite complex, but it can be simplified in the asymptotic case.

For Taylor's frozen-in hypothesis to work, we assume that $\mathbf{Z}_0^{\mp} \cdot \mathbf{k} \gg \nu^{\pm}(k)k^2$ and $\mathbf{Z}_0^{\mp} \cdot \mathbf{k} \gg k\tilde{U}_0(k)$. In addition, for $C^+(\tau)$ and $C^-(\tau)$, we choose the z axis to be along the direction of $\mathbf{Z}_0^-$ and $\mathbf{Z}_0^+$ respectively. For simplification, we make a change of variable, $\tilde{k}_{\pm} = Z_0^{\mp}k\tau$, and use $\epsilon = U^2/T$ (U is the rms speed). As a result, we obtain

$$C^{\pm}(\tau) \approx K^{\pm}(Z_0^{\mp}\tau)^{2/3}\frac{(\epsilon^{\pm})^{4/3}}{(\epsilon^{\mp})^{2/3}}\int d\tilde{k}_{\pm}\tilde{k}_{\pm}^{-5/3}f_L[\tilde{k}_{\pm}(L/Z_0^{\mp}\tau)]f_{\eta}[\tilde{k}_{\pm}(\eta/U_0\tau)]\frac{\sin(Z_0^{\mp}k\tau)}{Z_0^{\mp}k\tau} \times$$
$$\exp[-\tilde{k}_{\pm}^{2/3}(U/Z_0^{\mp})^{2/3}(\alpha^{\pm}\tau/T)^{1/3} - \tilde{k}_{\pm}^{4/3}(U/Z_0^{\mp})^{4/3}(\tau/T)^{2/3}], \tag{35}$$

where $\epsilon^{\pm} = \epsilon\alpha^{\pm}$. For τ in the inertial range, $L/Z_0^{\mp}\tau \gg 1$ and $\eta/Z_0^{\mp}\tau \ll 1$. Consequently, $f_L(\tilde{k}_{\pm}(L/Z_0^{\mp}\tau)) \approx 1$, and $f_{\eta}(\tilde{k}_{\pm}(\eta/Z_0^{\mp}\tau) \approx 1$. Therefore,

$$C^{\pm}(\tau) \approx K^{\pm}(Z_0^{\mp}\tau)^{2/3}\frac{(\epsilon^{\pm})^{4/3}}{(\epsilon^{\mp})^{2/3}}\int d\tilde{k}_{\pm}\tilde{k}_{\pm}^{-5/3}\frac{\sin\tilde{k}_{\pm}}{\tilde{k}_{\pm}} \times$$
$$\exp[-\tilde{k}_{\pm}^{2/3}(U/Z_0^{\mp})^{2/3}(\alpha^{\pm}\tau/T)^{1/3} - \tilde{k}_{\pm}^{4/3}(U/Z_0^{\mp})^{4/3}(\tau/T)^{2/3}]$$
$$= A^{\pm}K^{\pm}(Z_0^{\mp}\tau)^{2/3}\frac{(\epsilon^{\pm})^{4/3}}{(\epsilon^{\mp})^{2/3}}, \tag{36}$$

where $A^{\pm}$ are the values of the nondimensional integrals. The Fourier transform of the above $C^{\pm}(\tau)$ yields the following frequency spectra:

$$E^{\pm}(f) \approx \int C^{\pm}(\tau)\exp(-i2\pi f\tau)d\tau = \int A^{\pm}K^{\pm}(Z_0^{\mp}\tau)^{2/3}\frac{(\epsilon^{\pm})^{4/3}}{(\epsilon^{\mp})^{2/3}}\exp(-i2\pi f\tau)d\tau$$
$$= A'^{\pm}(|\mathbf{U}_0 \mp \mathbf{B}_0|)^{2/3}\frac{(\epsilon^{\pm})^{4/3}}{(\epsilon^{\mp})^{2/3}}f^{-5/3}, \tag{37}$$

where $A'^{\pm}$ are constants. Thus, we obtain $-5/3$ frequency spectra for Kolmogorov-like MHD turbulence phenomenology. Note that $E^+(f)$ and $E^-(f)$ are functions of $|\mathbf{U}_0 - \mathbf{B}_0|$ and $|\mathbf{U}_0 + \mathbf{B}_0|$, respectively, which are the respective speeds of $\mathbf{z}^+(\mathbf{k})$ and $\mathbf{z}^-(\mathbf{k})$ in the linear approximation. In comparison to Eq. (2), $E^{\pm}(f)$ has $U_0 \rightarrow |\mathbf{U}_0 \mp \mathbf{B}_0|$ and $\epsilon_u \rightarrow (\epsilon^{\pm})^{4/3}(\epsilon^{\mp})^{-2/3}$.

The calculation for the anisotropic MHD turbulence is more complex. However, the complexity is likely to be in the integral computation, which will reflect in the constants $A^{\pm}$. For anisotropic MHD turbulence, we expect that the form of $E^{\pm}(f)$ will the same as in Eq. (37).

The above analysis can be extended to Iroshnikov-Kraichnan phenomenology, where the correlation functions are

$$C^{\pm}(\mathbf{k}) = \frac{E^{\pm}(k)}{2\pi k^2} \sim (\epsilon B_0)^{1/2}k^{-7/2}. \tag{38}$$

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We substitute Eq. (38) into Eq. (30), employ $\nu^\pm(k)$ of Eq. (27), and set $\mathbf{r} = 0$. Following the same steps as above, we obtain

$$C^\pm(\tau) \approx (\epsilon B_0 Z_0^\mp \tau)^{1/2}. \tag{39}$$

Fourier transform of the above $C^\pm(\tau)$ yields the following frequency spectra:

$$E^\pm(f) = A_{\mathrm{IK}}(\epsilon B_0 |\mathbf{U_0} \mp \mathbf{B_0}|)^{1/2} f^{-3/2}, \tag{40}$$

where A_{IK} is a constant. Thus, we obtain $f^{-3/2}$ frequency spectrum for the Iroshnikov-Kraichnan phenomenology.

The prefactors for $E^\pm(f)$ are functions of $Z_0^\pm$. However, the prefactors for $E_u(k)$ and $E_b(k)$ would be more complex because

$$E^\pm(f) = E_u(f) + E_b(f) \pm 2H_c(f), \tag{41}$$

where $H_c = (1/2)\langle \mathbf{u} \cdot \mathbf{b} \rangle$. Clearly, derivation of $E_u(k)$, $E_b(k)$, and $H_c(k)$ requires further inputs, e.g., relationships among these functions. Based on these complexities,

V. DISCUSSION AND CONCLUSIONS

In this paper, we extend Taylor's frozen-in hypothesis to MHD turbulence. From the first principle, we derive one-point two-time correlation functions for MHD turbulence, whose Fourier transform yields the corresponding frequency spectra. The main predictions of our quantitative calculations are as follows:

1. The spectral indices for $E^\pm(k)$ and $E^\pm(f)$ are the same.

2. The prefactors of $E^\pm(f)$ are proportional to $|\mathbf{U_0} \mp \mathbf{B_0}|^{2/3}$ in Kolmogorov-like phenomenology, but proportional to $B_0^{1/2} |\mathbf{U_0} \mp \mathbf{B_0}|^{1/2}$ in Iroshnikov-Kraichnan phenomenology. In contrast to $E_u(f)$ for hydrodynamic turbulence, $U_0 \to |\mathbf{U_0} \mp \mathbf{B_0}|$ in $E^\pm(f)$ of MHD turbulence.

3. The kinetic and magnetic energy spectra, $E_u(f)$ and $E_b(f)$, are expected to have more complex prefactors.

4. When $B_0 \ll U_0$, the frequency spectrum of Eq. (2) can be employed for all the fields.

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The above predictions are important for the time series analysis of the solar wind and solar corona when U_0 and B_0 are comparable, e.g., for Parker Solar Probe (PSP) when it is close to the Sun. Hence, PSP's data provides an unique opportunity for testing the above predictions. We hope to validate these predictions in near future.

Solar wind data reveal another interesting property. Many authors[8,9,12] observe that the kinetic energy spectrum $(E_u(k))$ is steeper than the magnetic energy spectrum $(E_b(k))$. This relative steepening of $E_u(k)$ with relative to $E_b(k)$ is attributed to the energy transfers from the kinetic energy to the magnetic energy[29,30]. Note that these energy transfers are critical for the magnetic field generation or dynamo. Interestingly, $E^{\pm}(k)$ do not suffer from such steepening due to an absence of cross transfer between $\mathbf{z}^+$ and $\mathbf{z}^-$ (see Verma[16,17]). This is another reason why $E^{\pm}(k)$ are more reliable energy spectra compared to $E_u(k)$ and $E_b(k)$. It will be interesting to quantitatively compare the solar wind observations with the theoretical predictions made in this paper.

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